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The Cauchy Problem for the Wave Equation Using the Decomposition Method

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Abstract—This article presents the Adomian decomposition series solution to the Cauchy problem for the wave equation, which is a much simpler solution than that given by the Poisson formula. Further, the method is made applicable to various types of boundary conditions. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Decomposition method, Wave equation.

1. INTRODUCTION

Consider the Cauchy problem for the wave equation in R^n, namely,

$$u_{tt} - \nabla^2 u = 0, \qquad (\underline{x}, t) \in R^n \times R_+, \tag{1}$$

$$u(\underline{x}, 0) = p(\underline{x}), \quad u_t(\underline{x}, 0) = q(\underline{x}), \qquad \underline{x} \in R^n, \tag{2}$$

where $p \in C^3(R^n)$ and $q \in C^2(R^n)$ are given data. Using a very technical proof, one can show that, for $n = 3$, the solution of problem (1) and (2) is given by the Poisson formula, [1, pp. 222–226],

$$u(\underline{x}, t) = \frac{t}{4\pi} \iint_{|\underline{\rho}|=1} q(\underline{x} + t\underline{\rho})\, dS + \frac{\partial}{\partial t}\left[\frac{t}{4\pi} \iint_{|\underline{\rho}|=1} p(\underline{x} + t\underline{\rho})\, dS\right], \tag{3}$$

where $|\underline{\rho}| = 1$ is the unit sphere in R^3. Unfortunately, the evaluation of the integrals in expression (3) is fairly lengthy, even when p and q are polynomials.

Here we give a simpler series solution to problem (1) and (2) based on the Adomian decomposition method [2], which, in addition, does not depend on the dimension n, whereas in most PDE books, the solution of problem (1) and (2) has been obtained in different ways for different values of n. For example, $n = 3$ by the method of spherical means, $n = 2$ by the Hadamard method of descent, and $n = 1$ by the method of characteristics. In particular, we retrieve Chen's power series formula [3], and furthermore, we generalise our method to deal with various types of boundary conditions when the Cauchy problem is formulated in a bounded domain.

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2. DECOMPOSITION METHOD

The Adomian decomposition method [2] gives the solution of problem (1) and (2) in the form

$$u(\underline{x},t) = \lim_{K\to\infty} \phi_K(\underline{x},t), \qquad \phi_K(\underline{x},t) = \sum_{k=0}^{K} u_k(\underline{x},t), \tag{4}$$

where

$$u_0(\underline{x},t) = p(\underline{x}) + tq(\underline{x}), \quad u_{k+1}(\underline{x},t) = L_{tt}^{-1}\nabla^2 u_k(\underline{x},t), \qquad k \geq 0, \tag{5}$$

where $L_{tt} = \frac{\partial^2}{\partial t^2}$ and $L_{tt}^{-1} = \int_0^t dt' \int_0^{t'} dt''$. Conditions for the convergence of the decomposition series (4) were investigated in [4]. Applying the recurrence relation (5), we readily obtain the formal solution

$$u(\underline{x},t) = \sum_{k=0}^{\infty} \left\{ \frac{t^{2k}}{(2k)!}\nabla^{2k} p(\underline{x}) + \frac{t^{2k+1}}{(2k+1)!}\nabla^{2k} q(\underline{x}) \right\}. \tag{6}$$

This solution has been obtained previously in [3] using a classical power series and where it has been shown that if p and q are analytic in a neighbourhood of the origin, then solution (6) is the unique analytical continuation to the Cauchy problem (1) and (2) in a neighbourhood of the origin. Of course, in practice, the Cauchy data p and q will rarely be analytical, but then the mollification method [5] may be employed to smooth the data prior to employing the series solution (4).

So far, the decomposition method has provided a very simple way to obtain a formal series solution to the Cauchy problem formulated in the unbounded domain R^n. However, if the Cauchy problem is now formulated in a bounded domain $\Omega \subset R^n$, then boundary conditions on $\partial\Omega$ need to be supplied. In this respect, the power series solution of [3] becomes impracticable, but the decomposition method modified accordingly in order to take into account, these boundary conditions are still feasible; see Section 3.

3. THE CAUCHY PROBLEM FOR THE WAVE EQUATION IN A BOUNDED DOMAIN

Let the Cauchy problem (1) and (2) be formulated for $\underline{x} \in \Omega \subset R^n$ bounded, and let us assume, for simplicity of presentation, that Ω is the parallelepiped $\Omega = [0,a_1] \times [0,a_2] \times \cdots \times [0,a_n]$. In what follows, we shall investigate Dirichlet, Neumann, and mixed boundary conditions for u on the boundary $\partial\Omega$. Let us assume that we prescribe D, M, and N to be Dirichlet, mixed, and Neumann boundary conditions, respectively, in the form

$$u(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n,t) = f_{0i}, \qquad u(x_1,\dots,x_{i-1},a_i,x_{i+1},\dots,x_n,t) = f_{ai}, \quad i = \overline{1,D}, \tag{7}$$

$$u(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n,t) = f_{0i}, \qquad \frac{\partial u}{\partial x_i}(x_1,\dots,x_{i-1},a_i,x_{i+1},\dots,x_n,t) = g_{ai}, \quad i = \overline{(D+1),(D+M)}, \tag{8}$$

$$\frac{\partial u}{\partial x_i}(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n,t) = g_{0i}, \qquad \frac{\partial u}{\partial x_i}(x_1,\dots,x_{i-1},a_i,x_{i+1},\dots,x_n,t) = g_{ai}, \quad i = \overline{(D+M+1),(D+M+N)}, \tag{9}$$

respectively, where $\underline{x} = (x_1,\dots,x_n)$ and $n = D + M + N$. Of course, either D, M, or N can be zero. Then by denoting $L_{x_i x_i} = \frac{\partial^2}{\partial x_i^2}$, equation (1) can be rewritten as

$$L_{tt} u = \sum_{i=1}^{n} L_{x_i x_i} u. \tag{10}$$

Based on the boundary conditions (7)–(9), the corresponding inverse operators have to be defined as

$$
\begin{aligned}
L_{x_i x_i}^{-1} &= \int_0^{x_i} dx_i' \int_0^{x_i'} dx_i'' - \frac{x_i}{a_i}\int_0^{a_i} dx_i' \int_0^{x_i'} dx_i'', && i=\overline{1,D},\\
L_{x_i x_i}^{-1} &= \int_0^{x_i} dx_i' \int_{a_i}^{x_i'} dx_i'', && i=\overline{(D+1),(D+M)},\\
L_{x_i x_i}^{-1} &= \int_0^{x_i} dx_i' \int_0^{x_i'} dx_i'' - \frac{x_i^2}{2a_i}\int_0^{a_i} dx_i', && i=\overline{(D+M+1),(D+M+N)}.
\end{aligned} \tag{11}
$$

The initial starting term also modifies as

$$
\begin{aligned}
u_0 &= \frac{1}{n+1}\left\{p+tq+\sum_{i=1}^{D}\left[f_{0i}+\frac{x_i}{a_i}\left(f_{a_i}-f_{0i}\right)\right]\right\}\\
&+\frac{1}{n+1}\left\{\sum_{i=D+1}^{D+M}\left(f_{0i}+x_i g_{a_i}\right)+\sum_{i=D+M+1}^{n}\left[x_i g_{0i}+\frac{x_i^2}{2a_i}\left(g_{ai}-g_{0i}\right)+c_i\right]\right\},
\end{aligned} \tag{12}
$$

where $c_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,t)$ are unknown functions to be determined by imposing, at sufficiently large values of K, that ϕ_K satisfies equation (1) and initial conditions (2). Taking the inverses in equation (10) and adding, we obtain the recurrence relation

$$
u_{n+1} = \frac{1}{n+1}\left\{L_{tt}^{-1}\nabla^2+\sum_{i=1}^{n}L_{x_i x_i}^{-1}\left(L_{tt}-\sum_{j\neq i}^{n}L_{x_j x_j}\right)\right\}u_n, \qquad n\geq 0. \tag{13}
$$

At this stage, it should be noted that a similar approach for defining expressions (11)–(13) was attempted in [6] for the one-dimensional Cauchy problem for the wave equation with Dirichlet boundary conditions, but unfortunately with no illustration, probably because their test example, namely, $u(x,t)=e^{-t^2}\sin(x)$, does not satisfy the wave equation, and therefore, series (4) cannot converge to that solution claimed in [6]. Further, Dirichlet boundary conditions for the Laplace equation were investigated in [7], but the definition of the initial starting term and the inverse operators was rather heuristic and test problem dependent. In our case, all these concepts, as given by equations (11)–(13), are properly defined to satisfy the prescribed boundary conditions (7)–(9) when $x_i=0$ and $x_i=a_i$ for $i=\overline{1,n}$. Next, we illustrate our procedure through a simple example.

EXAMPLE. Let us take $u(x_1,x_2,x_3,t)=x_1^2+x_2^2+x_3^2+3t^2$ which satisfies wave equation (1) in the three-dimensional parallelepiped $\Omega=[0,a_1]\times[0,a_2]\times[0,a_3]\times\subset R^3$ and consider $D=M=N=1$, i.e., all three cases of possible boundary conditions are considered, as given by

$$
\begin{aligned}
f_{01} &= x_2^2+x_3^2+3t^2, \qquad f_{a1}=a_1^2+x_2^2+x_3^2+3t^2,\\
f_{02} &= x_1^2+x_3^2+3t^2, \qquad g_{a2}=2a_2, \qquad g_{03}=0, \qquad g_{a3}=2a_3.
\end{aligned} \tag{14}
$$

We also have the initial conditions (2) given by $p=x_1^2+x_2^2+x_3^2$ and $q=0$. Then the initial starting term (12) and the first term in recurrence relation (13) are given by

$$
\begin{aligned}
u_0 &= \frac{1}{4}\left[2x_1^2+2x_2^2+4x_3^2+6t^2+a_1x_1+2a_2x_2+c(x_1,x_2,t)\right],\\
u_1 &= \frac{1}{16}\left[L_{tt}^{-1}\left(16+\nabla^2 c\right)+L_{x_1x_1}^{-1}\left(\frac{\partial^2 c}{\partial t^2}-\frac{\partial^2 c}{\partial x_2^2}\right)+L_{x_2x_2}^{-1}\left(\frac{\partial^2 c}{\partial t^2}-\frac{\partial^2 c}{\partial x_1^2}\right)\right].
\end{aligned} \tag{15}
$$

From equation (15), it can be seen that due to the presence of the unknown function $c(x_1,x_2,t)$, the expressions for u_n, $n\geq 2$, get more and more complicated, and at this stage, it is sufficient to remark that u_n does not depend on x_3 for all $n \geq 1$. Therefore, we can reduce the dimensionality of the problem by one by defining a new function $v(x_1, x_2, t) = u(x_1, x_2, x_3, t) - x_3^2 - t^2$, which will satisfy the following reduced problem:

$$\begin{aligned}
\nabla^2 v &= \frac{\partial^2 v}{\partial t^2}, & (x_1, x_2, t) &\in [0, a_1] \times [0, a_2] \times R_+, \\
v(x_1, x_2, 0) &= \bar{p} = x_1^2 + x_2^2, & v_t(x_1, x_2, 0) &= \bar{q} = 0, \\
v(0, x_2, t) &= \bar{f}_{01} = x_2^2 + 2t^2, & v(a_1, x_2, t) &= \bar{f}_{a1} = a_1^2 + x_2^2 + 2t^2, \\
v(x_1, 0, t) &= \bar{f}_{02} = x_1^2 + 2t^2, & \frac{\partial v}{\partial x_2}(x_1, a_2, t) &= \bar{g}_{a2} = 2a_2.
\end{aligned} \tag{16}$$

It should be noted that if $N = 0$, this elimination is not necessary. But if $N \geq 2$, the elimination is not so obvious and it should be investigated in a future work.

Then the Adomian decomposition method for problem (16) suggests defining the initial starting term and the recurrence relation, as follows:

$$\begin{aligned}
v_0 &= \frac{1}{3}\left[\bar{p} + t\bar{q} + \bar{f}_{01} + \frac{x_1}{a_1}\left(\bar{f}_{a1} - \bar{f}_{01}\right) + \bar{f}_{02} + x_2 \bar{g}_{a2}\right] \\
&= \frac{1}{3}\left[2x_1^2 + 2x_2^2 + 4t^2 + a_1 x_1 + 2a_2 x_2\right],
\end{aligned} \tag{17}$$

$$v_{n+1} = \frac{1}{3}\left[L_{tt}^{-1}\left(L_{x_1x_1} + L_{x_2x_2}\right) + L_{x_1x_1}^{-1}\left(L_{tt} - L_{x_2x_2}\right) + L_{x_2x_2}^{-1}\left(L_{tt} - L_{x_1x_1}\right)\right] v_n, \qquad n \geq 0. \tag{18}$$

Applying the recurrence relation (18) with the starting term (17), we immediately obtain that

$$v_n = \frac{1}{3^{n+1}}\left[2x_1^2 + 2x_2^2 + 4t^2 - 2a_1 x_1 - 4a_2 x_2\right], \qquad n \geq 1. \tag{19}$$

Thus, the series solution (4) for v is given by

$$\begin{aligned}
v = \sum_{n=0}^{\infty} v_n &= \left(2x_1^2 + 2x_2^2 + 4t^2\right)\left(-1 + \sum_{n=0}^{\infty} 3^{-n}\right) \\
&+ \frac{(a_1 x_1 + 2a_2 x_2)}{3}\left(1 - 2\sum_{n=0}^{\infty} 3^{-n-1}\right) = x_1^2 + x_2^2 + 2t^2.
\end{aligned} \tag{20}$$

Hence, $u(x_1, x_2, x_3, t) = v(x_1, x_2, t) + x_3^2 + t^2 = x_1^2 + x_2^2 + x_3^2 + 3t^2$, as required.

For more complicated examples, specialised software manipulations such as MAPLE should be used.

4. CONCLUSIONS

In this paper, the Adomian decomposition method has been applied for solving Cauchy problems for the wave equation. For an unbounded domain in any dimension, we recover the power series formula of [3], which is much simpler than the classical Poisson formula. Further, we accommodate the decomposition method to deal with bounded domains, and Dirichlet, Neumann, and mixed boundary conditions are all investigated. In comparison with the previous works on the decomposition method, we have used definite integral operators and starting terms which incorporate in a natural manner the initial and boundary conditions and which in turn always provide a convergent numerical solution to the correct limit.

Further, the same analysis can also be applied to Cauchy problems for the heat equations which have been investigated elsewhere [8]. Finally, extensions of the method to higher dimensions, inhomogeneous, and nonlinear situations can be accommodated using the techniques of [3,9].

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